

# On Rényi and Tsallis entropies and divergences for exponential families

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**Abstract**—Many common probability distributions in statistics like the Gaussian, multinomial, Beta or Gamma distributions can be studied under the unified framework of exponential families. In this paper, we prove that both Rényi and Tsallis divergences of distributions belonging to the same exponential family admit a generic closed form expression. Furthermore, we show that Rényi and Tsallis entropies can also be calculated in closed-form for sub-families including the Gaussian or exponential distributions, among others.

**Index Terms**—Shannon entropy, Rényi entropies ; Tsallis entropies ; divergences; exponential families.

## I. INTRODUCTION

In 1948, Shannon published a theory on communications that initiated the field of information theory [1]. Nowadays, it is well-known that Shannon *entropy* quantitatively measures the amount of *uncertainty* [2] of a random variable. The entropy  $H(P)$  of a random variable  $P$  is defined according to its underlying density  $p(x)$  as

$$\begin{aligned} H(P) &= \int p(x) \log \frac{1}{p(x)} dx \\ &= - \int p(x) \log p(x) dx = E_P[-\log p(x)]. \end{aligned} \quad (1)$$

Closed-form expressions for the Shannon entropy for many continuous distributions are reported in [3]. In coding theory, one seeks for codes that uses the underlying structure of the message language at its best. Since in practice, the true model distribution  $P$  is hidden by nature and therefore unknown to the observer, we rather define the *cross-entropy* between the considered model  $Q$  and the unknown ideal random variables  $P$  as

$$\begin{aligned} H^\times(P : Q) &= E_P[-\log q(x)] = - \int p(x) \log q(x) dx \\ &\geq H^\times(P : P). \end{aligned} \quad (4)$$

It follows that the Kullback-Leibler divergence (also called relative entropy) between two distributions  $P$  and  $Q$  is defined by

$$KL(P : Q) = \int_x p(x) \log \frac{p(x)}{q(x)} dx = E_P \left[ \log \frac{p(x)}{q(x)} \right] \quad (5)$$

The Kullback-Leibler divergence  $KL(P : Q)$  is an oriented distance (i.e.,  $KL(P : Q) \neq KL(Q : P)$ , emphasized by the “:” notational convention) that can be rewritten as

$$KL(P : Q) = H^\times(P : Q) - H(P) \geq 0 \quad (6)$$

In 1961, Rényi generalized the Shannon entropy by modifying one of its axiom characterizing the *averaging* of information. The Rényi  $H_\alpha(p)$  entropy [4] of a probability distribution  $p$  is a single-parametric function defined by

$$H_\alpha^R(p) = \frac{\log \int p^\alpha(x) dx}{1 - \alpha}, \alpha \in (0, +\infty) \setminus \{1\}. \quad (7)$$

Let us prove using L'Hôpital rule<sup>1</sup> that Rényi entropies tend to Shannon entropy  $H(p) = - \int p(x) \log p(x) dx$  when  $\alpha \rightarrow 1$ . (This is a classical proof explained in textbooks that we include here to illustrate L'Hôpital rule that we shall use repeatedly later on.)

*Proof:* Consider the discrete case (i.e., counting measure) of Rényi and Shannon entropies. Set  $f(\alpha) = \log \sum_{i=1}^n p_i^\alpha$  (for any fixed distribution  $P$ ) and  $g(\alpha) = 1 - \alpha$ . Then  $\frac{df(\alpha)}{d\alpha} = -1$  and

$$\frac{df(\alpha)}{d\alpha} = \frac{\sum_{i=1}^n \frac{d}{d\alpha} (p_i^\alpha)}{\sum_{i=1}^n p_i^\alpha} \quad (8)$$

after applying the derivative chain rule. Since

$$\frac{d}{d\alpha} (p_i^\alpha) = \frac{d}{d\alpha} e^{\alpha \log p_i} = (\log p_i) e^{\alpha \log p_i} = p_i^\alpha \log p_i, \quad (9)$$

we get

$$\frac{f'(\alpha)}{g'(\alpha)} = - \sum_{i=1}^n p_i^\alpha \log p_i, \text{ and } \lim_{\alpha \rightarrow 1} \frac{f'(\alpha)}{g'(\alpha)} = - \sum_{i=1}^n p_i \log p_i. \quad (10)$$

Since  $\lim_{\alpha \rightarrow 1} f(\alpha) = \lim_{\alpha \rightarrow 1} g(\alpha) = 0$  and  $\lim_{\alpha \rightarrow 1} \frac{f'(\alpha)}{g'(\alpha)} = - \sum_{i=1}^n p_i \log p_i$ , we deduce from l'Hôpital rule that  $\lim_{\alpha \rightarrow 1} H_\alpha^R(P) = H(P)$ . That is, Rényi entropy tends to Shannon entropy as  $\alpha \rightarrow 1$ . ■

Rényi entropies keep Shannon additivity property [2] of independent systems, and are concave and monotonically decreasing function of  $\alpha$ . Closed-form formula for the Rényi entropies of many multivariate distributions are reported in [5], and for the multivariate Gaussian distribution in the technical report [6].

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<sup>1</sup>L'Hôpital rule dates back to the 17th century, and states that the limit of the indeterminate ratio of functions equals to the limit of the ratio of their derivatives provided that (i) the limits of both the numerator and denominator coincide, and that (ii) the limit of the ratio of the derivatives also exists. That is, if  $\lim_{x \rightarrow \alpha} f(x) = \lim_{x \rightarrow \alpha} g(x) = 0$  and  $\lim_{x \rightarrow \alpha} f'(x)/g'(x) = l$  exists, then  $\lim_{x \rightarrow \alpha} \frac{f(x)}{g(x)} = \lim_{x \rightarrow \alpha} \frac{f'(x)}{g'(x)} = l$ .

In 1988, Tsallis (motivated by physical multi-fractal systems) introduced yet another one-parameter generalization of Shannon entropy. Historically, this family of entropic functions was derived axiomatically by Havrda and Charvat [7] in 1967. Tsallis  $H_\alpha^T(p)$  entropies of a probability distribution  $p$  are defined by

$$H_\alpha^T(p) = \frac{\int p(x)^\alpha dx - 1}{1 - \alpha}, \alpha \in \mathbb{R} \setminus \{1\} \quad (11)$$

Tsallis entropies are non-additive, tending to Shannon entropy when  $\alpha \rightarrow 1$ , and can be derived from the generalized Shannon-Khinchin axioms [8].

Let  $I_\alpha(p) = \int p(x)^\alpha dx$ , then Rényi and Tsallis entropies can be rewritten as

$$H_\alpha^R(p) = \frac{\log I_\alpha(p)}{1 - \alpha}, \quad (12)$$

$$H_\alpha^T(p) = \frac{I_\alpha(p) - 1}{1 - \alpha}. \quad (13)$$

Since  $I_\alpha(p) = (1 - \alpha)H_\alpha^T(p) + 1 = e^{(1-\alpha)H_\alpha^R(p)}$ , we can convert these two families of entropies through the following monotonic conversion functions:

$$H_\alpha^R(p) = \frac{\log((1 - \alpha)H_\alpha^T(p) + 1)}{1 - \alpha} \quad (14)$$

$$H_\alpha^T(p) = \frac{e^{(1-\alpha)H_\alpha^R(p)} - 1}{1 - \alpha} \quad (15)$$

## II. RÉNYI AND TSALLIS ENTROPIES OF EXPONENTIAL FAMILIES

A random variable  $X \sim E_F(\theta)$  is said to belong to the exponential family  $E_F$  [9] when it admits the following canonical decomposition of its density:

$$p_F(x; \theta) = \exp(\langle t(x), \theta \rangle - F(\theta) + k(x)), \quad (16)$$

where  $\langle x, y \rangle = x^T y$  denotes the inner product,  $t(x)$  the sufficient statistics,  $\theta$  the natural parameters,  $F(\theta)$  a  $C^\infty$  differentiable real-valued convex function, and  $k(x)$  a carrier measure.

Since  $F(\theta) = \log \int_x \exp(\langle t(x), \theta \rangle + k(x)) dx$  (because  $\int p_F(x; \theta) dx = 1$ ), function  $F$  is called the log-normalizer. Function  $F$  characterizes the family, while the natural parameter  $\theta$  denotes the member of the family  $E_F$ .

A statistic is a function of the observations (say, the sample mean or sample variance) that collects information about the distribution with the goal to concentrate information for later inference. A statistic is said sufficient if it allows one to concentrate information obtained from random observations *without losing* information, in a sense that working directly on the observation sets or its compact sufficient statistics yields exactly the same parameter estimation results. It can be shown from the Neyman-Pearson factorization theorem [2], under mild regularity conditions, that the class of distributions admitting sufficient statistics are precisely the exponential families [10]. Term  $k(x)$  is related to the carrier measure (i.e.,

counting or Lebesgue). An exponential family can be univariate (eg., like the Poisson or 1D Gaussian distributions) or multivariate (like the multinomial or  $d$ -dimensional Gaussian distributions). The order of an exponential family denotes the dimension of the parameter space. Thus the Gaussian (normal) distribution is univariate of order 2 (parameters  $\mu$  and  $\sigma$ ).

Many common distribution families such as Poisson, Gaussian or multinomial distributions are exponential families whose canonical decompositions  $(F, t, \theta, k)$  are given in [10].

Let us prove that for any distribution belonging to the exponential families, we have the following entropy expressions:

$$\begin{aligned} H_\alpha^R(p_F(x; \theta)) &= \frac{1}{1 - \alpha} \left( F(\alpha\theta) - \alpha F(\theta) + \log E_p[e^{(\alpha-1)k(x)}] \right) \end{aligned} \quad (17)$$

$$\begin{aligned} H_\alpha^T(p_F(x; \theta)) &= \frac{1}{1 - \alpha} \left( (e^{F(\alpha\theta) - \alpha F(\theta)}) E_p[e^{(\alpha-1)k(x)}] - 1 \right) \end{aligned} \quad (18)$$

*Proof:*

Consider calculating  $I_\alpha(p) = \int p(x)^\alpha dx$  term for exponential families:

$$I_\alpha(p) = \int e^{\alpha(\langle t(x), \theta \rangle - F(\theta) + k(x))} dx \quad (19)$$

$$= \int e^{\langle t(x), \alpha\theta \rangle - \alpha F(\theta) + \alpha k(x) + (1-\alpha)k(x) - (1-\alpha)k(x) + F(\alpha\theta) - F(\alpha\theta)} dx \quad (20)$$

$$= \int e^{F(\alpha\theta) - \alpha F(\theta)} p_F(x; \alpha\theta) e^{(\alpha-1)k(x)} dx \quad (21)$$

$$= e^{F(\alpha\theta) - \alpha F(\theta)} \int p_F(x; \alpha\theta) e^{(\alpha-1)k(x)} dx \quad (22)$$

$$= e^{F(\alpha\theta) - \alpha F(\theta)} E_p[e^{(\alpha-1)k(x)}] \quad (23)$$

The formula for the Rényi and Tsallis entropies are then derived using Eq. 12 and Eq. 13. ■

In particular, for standard carrier measure  $k(x) = 0$  (eg., Gaussian, exponential, Bernoulli or centered Laplacian), we obtain the following generic *closed-form* expressions of Rényi and Tsallis entropies:

$$H_\alpha^R(p_F(x; \theta)) = \frac{1}{1 - \alpha} (F(\alpha\theta) - \alpha F(\theta)) \quad (24)$$

$$H_\alpha^T(p_F(x; \theta)) = \frac{1}{1 - \alpha} (e^{F(\alpha\theta) - \alpha F(\theta)} - 1) \quad (25)$$

For  $\alpha \rightarrow 1$ , observe that both those formula yields to Shannon entropy for exponential families (with  $k(x) = 0$ ):

$$H(p_F(x; \theta)) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle \quad (26)$$

*Proof:* Let us use L'Hôpital rule on Rényi entropy

$$H_\alpha(p_F(x; \theta)) = \frac{1}{1 - \alpha} (F(\alpha\theta) - \alpha F(\theta)) \quad (27)$$

$$\simeq_{\alpha \rightarrow 1} \frac{\langle \theta, \nabla F(\alpha\theta) \rangle - F(\theta)}{-1} \quad (28)$$

$$\simeq_{\alpha \rightarrow 1} F(\theta) - \langle \theta, \nabla F(\theta) \rangle \quad (29)$$

(using Gâteaux derivatives  $\nabla_\alpha F(\alpha\theta) = \langle \theta, \nabla F(\alpha\theta) \rangle$ ) ■

For non-zero carrier measure the Shannon entropy of an exponential family  $p \sim E_F(\theta)$  is  $H(p_F(x; \theta)) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle - E_p[k(x)]$ . This will be proved in section III.

**Example** To illustrate the generic entropy formula, let us start with a simple exponential family: the exponential distribution. The exponential distribution models the time between two successive Poisson processes, and has density

$$p(x; \lambda) = \lambda e^{-\lambda x}, x \geq 0 \quad (30)$$

where  $\lambda > 0$  is called the rate parameter.

Writing  $\lambda e^{-\lambda x} = e^{-\lambda x + \log \lambda}$ , we get the canonical decomposition of exponential families with  $t(x) = x$ ,  $\theta = -\lambda$ ,  $F(\theta) = -\log \lambda = -\log -\theta$  and  $k(x) = 0$ . The exponential distribution is a univariate exponential family of order 1. The Rényi entropy is  $H_\alpha^R(p) = \frac{1}{1-\alpha}(F(\alpha\theta) - \alpha F(\theta)) = \frac{1}{1-\alpha}(-\log \alpha\lambda + \alpha \log \lambda) = \log \lambda - \frac{\log \alpha}{1-\alpha}$ . The log-normalizer derivative is  $F'(\theta) = -\frac{1}{\theta} = \frac{1}{\lambda}$ . The Shannon entropy is  $H(p) = F(\theta) - \theta F'(\theta) = 1 - \log \lambda$ . Using L'Hôpital rule, we find that  $\lim_{\alpha \rightarrow 1} H_\alpha^R(p) = -\log \lambda = H(p)$ . Tsallis entropy is  $\frac{1}{1-\alpha}(\frac{\lambda^\alpha}{\alpha\lambda} - 1) = \frac{\lambda^\alpha - \alpha\lambda}{\alpha(1-\alpha)\lambda}$ . Again, using L'Hôpital rule, we find that Tsallis entropy converges to Shannon entropy as  $\alpha \rightarrow 1$ :  $H_\alpha^T(p) \lim_{\alpha \rightarrow 1} \frac{(\lambda^\alpha - \alpha\lambda)'}{(\alpha(1-\alpha)\lambda)'} = \frac{\lambda \log \lambda - \lambda}{\lambda} = 1 - \log \lambda = H(p)$  (where the derivatives are computed according to parameter  $\alpha$ ). ◇

**Example** Let us consider now the usual Gaussian distribution (univariate of order 2) with density

$$p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \quad (31)$$

Its canonical decomposition into an exponential family yields

$$p(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad (32)$$

$$= \exp\left(-\frac{x^2}{2\sigma^2} + x\frac{\mu}{\sigma^2} - \frac{\mu^2}{2\sigma^2} - \frac{1}{2}\log 2\pi\sigma^2\right) \quad (33)$$

$$= \exp\left(\left\langle \begin{pmatrix} x \\ x^2 \end{pmatrix}, \begin{pmatrix} \frac{\mu}{\sigma^2} \\ -\frac{1}{2\sigma^2} \end{pmatrix} \right\rangle - F(\theta)\right) \quad (34)$$

- Sufficient statistics:  $t(x) = (x, x^2)$ ,
- Natural parameters  $\theta = (\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2})$ ,
- Log-normalizer  $F(\theta) = -\frac{\theta_1^2}{4\theta_2} + \frac{1}{2}\log \frac{2\pi}{-\theta_2} = \frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log 2\pi\sigma^2$ , and
- Carrier measure  $k(x) = 0$ .

Thus the Rényi entropy of Eq. 24 instantiated to the Gaussian case is

$$H_\alpha^R(p) = \frac{1}{1-\alpha}(F(\alpha\theta) - \alpha F(\theta)) \quad (35)$$

$$= \frac{1}{1-\alpha}\left(\alpha\frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log \frac{2\pi\sigma^2}{\alpha} - \alpha\left(\frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log 2\pi\sigma^2\right)\right) \quad (36)$$

$$= \frac{1}{1-\alpha}\left(\frac{1-\alpha}{2}\log 2\pi\sigma^2 - \frac{1}{2}\log \alpha\right) \quad (37)$$

$$= \frac{1}{2}\log 2\pi\sigma^2 - \frac{\log \alpha}{2(1-\alpha)} \quad (38)$$

When  $\alpha \rightarrow 1$ ,  $H_\alpha^R(p) \rightarrow \frac{1}{2}\log 2\pi e\sigma^2$  (using L'Hôpital rule  $\frac{\log \alpha}{2(\alpha-1)} \simeq_{\alpha \rightarrow 1} -\frac{1}{2}$ ). Now using the Shannon closed form entropy of Eq. 26 with  $\nabla F(\theta) = (-\frac{\theta_1}{2\theta_2}, \frac{\theta_1^2}{4\theta_2} - \frac{1}{2\theta_2}) = (\mu, \mu^2 + \sigma^2)$ , we again find the Gaussian entropy  $H(p) = \frac{1}{2}\log 2\pi e\sigma^2$ :

$$H(p) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle \quad (39)$$

$$= \frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log 2\pi\sigma^2 - \left\langle \begin{pmatrix} \mu \\ \mu^2 + \sigma^2 \end{pmatrix}, \begin{pmatrix} \mu \\ \mu^2 + \sigma^2 \end{pmatrix} \right\rangle \quad (40)$$

$$= \frac{\mu^2}{2\sigma^2} + \frac{1}{2}\log 2\pi\sigma^2 - \frac{\mu^2}{\sigma^2} + \frac{\mu^2}{2\sigma^2} + \frac{1}{2} \quad (41)$$

$$= \frac{1}{2}\log 2\pi e\sigma^2 \quad (42)$$

It follows from the conversion formula of Eq. 15 that the Tsallis entropy of the Gaussian is

$$H_\alpha^T(p) = \frac{e^{(1-\alpha)\log(2\pi e\sigma^2)^{\frac{1}{2}}} - 1}{1-\alpha} = \frac{(2\pi e\sigma^2)^{\frac{1-\alpha}{2}}}{1-\alpha}. \quad (43)$$

Again, we check that when  $\alpha \rightarrow 1$ , the Tsallis entropy tends to Shannon entropy:

$$\frac{e^{(1-\alpha)\log(2\pi e\sigma^2)^{\frac{1}{2}}} - 1}{1-\alpha} \simeq_{\alpha \rightarrow 1} \frac{1 + (1-\alpha)\log(2\pi e\sigma^2)^{\frac{1}{2}} - 1}{1-\alpha} \quad (44)$$

$$= \frac{1}{2}\log 2\pi e\sigma^2 \quad (45)$$

◇

Similarly but with greater matrix calculus complexity, based on the canonical decomposition reported in [10], we may consider the multivariate Gaussian distribution  $X \sim \mathcal{N}(\mu, \Sigma)$  with mean  $\mu$  and covariance matrix  $\Sigma$  ( $\det \Sigma = |\Sigma| > 0$ ). Appendix A provides the calculus details. We report the result here. The Rényi  $\alpha$ -entropy is given by

$$H_\alpha^R(X) = \frac{d}{2}\log 2\pi + \frac{1}{2}\log |\Sigma| + \frac{d\log \alpha}{2(\alpha-1)} \quad (46)$$

and tend to Shannon entropy as  $\alpha \rightarrow 1$  (using L'Hôpital rule  $\frac{d\log \alpha}{2(\alpha-1)} \simeq_{\alpha \rightarrow 1} \frac{d}{2}$ ):

$$H(X) = \frac{1}{2}\log(2\pi e)^d |\Sigma| \quad (47)$$

Note that the sufficient statistics  $t(x)$  does not intervene in the entropy formula. The sufficient statistics plays a role for estimating parameter  $\theta$  from independent and identically distributed (i.i.d.) observations, as mentioned in the concluding remarks (see Section V).

### III. RÉNYI AND TSALLIS DIVERGENCES OF EXPONENTIAL FAMILIES

Consider now two probability distributions  $P$  and  $Q$ , and define the Rényi  $D_\alpha^R(p : q)$  and Tsallis  $D_\alpha^T(p : q)$  divergences as follows

$$D_\alpha^R(p : q) = \frac{\log \int p(x)^\alpha q(x)^{1-\alpha} dx}{\alpha - 1} \quad (48)$$

$$D_\alpha^T(p : q) = \frac{\int p(x)^\alpha q(x)^{1-\alpha} dx - 1}{\alpha - 1} \quad (49)$$

Those divergences are related to the  $\alpha$ -divergence<sup>2</sup>

$$I_\alpha(p : q) = \int p(x)^\alpha q(x)^{1-\alpha} dx \quad (50)$$

that plays an important role<sup>3</sup> in information geometry [12]:

$$D_\alpha^R(p : q) = \frac{\log I_\alpha(p : q)}{\alpha - 1} \quad (51)$$

$$D_\alpha^T(p : q) = \frac{I_\alpha(p : q) - 1}{\alpha - 1} \quad (52)$$

Rényi divergence can also be rewritten as

$$D_\alpha^R(p : q) = \frac{\log \int p(x)^\alpha q(x)^{1-\alpha} dx}{\alpha - 1} = \frac{\log \int \left( \frac{p(x)}{q(x)} \right)^\alpha q(x) dx}{\alpha - 1}, \quad (53)$$

that shows it is a Csiszár  $f$ -divergence [13]. The special case  $\alpha = \frac{1}{2}$  yields

$$D_{\frac{1}{2}}^R(p : q) = -2 \log \int \sqrt{p(x)} \sqrt{q(x)} dx = -2 \log B(p, q), \quad (54)$$

where  $B(p, q) = \int \sqrt{p(x)} \sqrt{q(x)} dx$  is called the Bhattacharyya coefficient [14]. The Bhattacharyya coefficient is itself related to the (squared) Hellinger distance [15]:

$$H^2(p : q) = \frac{1}{2} \int (\sqrt{p(x)} - \sqrt{q(x)})^2 dx \quad (55)$$

$$= \frac{1}{2} \left( \int p(x) dx + \int q(x) dx - 2 \int (\sqrt{p(x)} \sqrt{q(x)}) dx \right) \quad (56)$$

$$= 1 - B(p, q). \quad (57)$$

For members of the same exponential families (we do not require standard carrier measure  $k(x)$  to be zero anymore), the Rényi and Tsallis divergences [16] can always be calculated from the following closed-form solution:

$$D_\alpha^R(p_F(x; \theta) : p_F(x; \theta')) = \frac{1}{1 - \alpha} J_{F, \alpha}(\theta : \theta') \quad (58)$$

$$D_\alpha^T(p_F(x; \theta) : p_F(x; \theta')) = \frac{1}{1 - \alpha} \left( e^{-J_{F, \alpha}(\theta : \theta')} - 1 \right) \quad (59)$$

where

$$J_{F, \alpha}(\theta : \theta') = \alpha F(\theta) + (1 - \alpha) F(\theta') - F(\alpha \theta + (1 - \alpha) \theta') \quad (60)$$

<sup>2</sup>Historically, this divergence was first presented by Chernoff in [11].

<sup>3</sup>Namely, the role of canonical divergence in constant curvature statistical manifolds.

is the skew divergence based on the Jensen gap obtained from the log-normalizer convex function  $F$ .  $J_{F, \alpha}(\theta : \theta')$  is non-negative for  $\alpha \in [0, 1]$  and non-positive for  $\alpha \in (-\infty, 0] \cup [1, \infty)$ . It loses discriminatory power (i.e.,  $J_F(\theta : \theta') = 0, \forall \theta, \theta'$ ) for  $\alpha \in \{0, 1\}$ .

*Proof:* Let us consider computing  $I_\alpha(p : q) = I_\alpha(\theta : \theta')$  for members  $p \sim E_F(\theta)$  and  $q \sim E_F(\theta')$  of the same exponential family  $E_F$ :

$$I_\alpha(p : q) = \int p(x)^\alpha q(x)^{1-\alpha} dx \quad (61)$$

$$I_\alpha(\theta : \theta') = \int \exp^{\alpha(\langle t(x), \theta \rangle - F(\theta) + k(x))} \quad (62)$$

$$\times \exp^{(1-\alpha)(\langle t(x), \theta' \rangle - F(\theta') + k(x))} dx \quad (63)$$

$$= \int e^{\langle t(x), \alpha \theta + (1-\alpha) \theta' \rangle} \quad (64)$$

$$\times \exp^{-\alpha F(\theta) - (1-\alpha) F(\theta') + k(x)} dx \quad (65)$$

$$= \int e^{F(\alpha \theta + (1-\alpha) \theta') - \alpha F(\theta)} \quad (66)$$

$$\times \exp^{-(1-\alpha) F(\theta') - p_F(x; \alpha \theta + (1-\alpha) \theta')} dx \quad (67)$$

$$= e^{-J_{F, \alpha}(\theta : \theta')} \underbrace{\int p_F(x; \alpha \theta + (1-\alpha) \theta') dx}_{=1} \quad (68)$$

$$= e^{-J_{F, \alpha}(\theta : \theta')} > 0 \quad (69)$$

Thus the Rényi divergence of members of the same exponential family amounts to compute a scaled skew Jensen divergence for the log-normalizer:

$$D_\alpha^R(p : q) = \frac{J_{F, \alpha}(\theta : \theta')}{1 - \alpha} \quad (70)$$

Note that for  $\alpha > 1$ , we have both  $1 - \alpha < 0$  and  $J_{F, \alpha}(\theta : \theta') < 0$  so that Rényi divergence is non-negative. (However, for  $\alpha < 0$ ,  $1 - \alpha > 0$  but  $J_{F, \alpha}(\theta : \theta') < 0$ . This shows that Rényi divergences are defined for  $\alpha \in (0, \infty) \setminus \{1\}$ .)

The formula for Tsallis divergence follows from the following conversion formula

$$D_\alpha^T(p : q) = \frac{e^{(\alpha-1)D_\alpha^R(p : q)} - 1}{\alpha - 1} = \frac{e^{-J_{F, \alpha}(\theta : \theta')} - 1}{\alpha - 1} \quad (71)$$

Observe that  $D_\alpha^T(p : q) \simeq_{\alpha \rightarrow 1} D_\alpha^R(p : q) \simeq_{\alpha \rightarrow 1} \text{KL}(p : q)$  (using the argument  $e^x \simeq_{x \rightarrow 0} 1 + x$ ). When  $\alpha \rightarrow 1$ , we get the well-known result [17], [18] that

$$\text{KL}(p_F(x; \theta) : p_F(x; \theta')) = B_F(\theta' : \theta), \quad (72)$$

where  $B_F$  is the Bregman divergence [19] defined by

$$B_F(p : q) = F(p) - F(q) - \langle p - q, \nabla F(q) \rangle. \quad (73)$$

*Proof:* Consider the limit case of Rényi divergence of members of the same exponential family as  $\alpha \rightarrow 1$ . We shall use the following Taylor expansion (Gâteaux derivative) of a skew Jensen divergence:

$$J_{F,\alpha}(\theta : \theta') = F(\alpha\theta + (1-\alpha)\theta') \quad (74)$$

$$-\alpha F(\theta) - (1-\alpha)F(\theta') \quad (75)$$

$$\simeq_{\alpha \rightarrow 1} (1-\alpha)F(\theta') + (1-\alpha)\langle \theta' - \theta, \nabla F(\theta) \rangle - (1-\alpha)F(\theta) \quad (76)$$

$$- (1-\alpha)F(\theta) \quad (77)$$

$$\lim_{\alpha \rightarrow 1} D_{\alpha}^R(p : q) = \lim_{\alpha \rightarrow 1} \frac{J_{F,\alpha}(\theta : \theta')}{1-\alpha} = \text{KL}(p : q) \quad (78)$$

$$\simeq_{\alpha \rightarrow 1} F(\theta') - F(\theta) - \langle \theta' - \theta, \nabla F(\theta) \rangle \quad (79)$$

$$= B_F(\theta' : \theta) \quad (80)$$

A direct alternative proof is also given in Appendix A.

**Example** Consider the exponential distribution. We recall that the natural parameter is  $\theta = -\lambda$  and the log-normalizer  $F(\theta) = -\log -\theta = -\log \lambda$ . For two members  $p \sim E_F(\theta)$  and  $q \sim E_F(\theta')$  of the same family of exponential distributions, we have the Rényi divergence  $D_{\alpha}^R(p : q) = \frac{1}{1-\alpha}(\alpha F(\theta) + (1-\alpha)F(\theta') - F(\alpha\theta + (1-\alpha)\theta')) = \frac{1}{1-\alpha} \log \frac{\alpha\lambda + (1-\alpha)\lambda'}{\lambda^{\alpha}\lambda'^{1-\alpha}}$ . The Tsallis divergence is  $D_{\alpha}^T(p : q) = \frac{1}{1-\alpha}(\frac{\lambda^{\alpha}\lambda'^{1-\alpha}}{\alpha\lambda + (1-\alpha)\lambda'} - 1)$ .  $\diamond$

#### IV. SHANNON ENTROPY AND CROSS-ENTROPY FOR THE EXPONENTIAL FAMILIES

Let us now prove that the Shannon entropy and cross-entropy of distributions belonging to the same exponential family can be expressed as

$$H(p) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle - E_{\theta}[k(x)] \quad (81)$$

$$H(p : q) = F(\theta') - \langle \theta', \nabla F(\theta) \rangle - E_{\theta}[k(x)] \quad (82)$$

*Proof:* Write the relative entropy as the difference of the cross-entropy minus the entropy:

$$\text{KL}(p : q) = H^{\times}(p : q) - H(p). \quad (83)$$

For distributions belonging to the same exponential families, we can separate the terms independent of  $q$  (i.e.,  $\theta'$ ) from the terms depending on  $p$  (i.e.,  $\theta$ ), to get

$$\begin{aligned} \text{KL}(p : q) &= B_F(\theta' : \theta) \\ &= F(\theta') - F(\theta) - \langle \theta' - \theta, \nabla F(\theta) \rangle \\ &= \underbrace{F(\theta') - \langle \theta', \nabla F(\theta) \rangle}_{\sim H_F^{\times}(\theta : \theta')} - \underbrace{(F(\theta) - \langle \theta, \nabla F(\theta) \rangle)}_{\sim H_F(\theta)} \end{aligned}$$

Since the Bregman convex generator  $F$  is defined up to an affine term  $ax + b$  in the Bregman divergence, and since the factor  $a$  leaves independent both the entropy and cross-entropy terms, we deduce that

$$H(p) = H_F(\theta) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle + b, \quad (84)$$

where  $b$  is a constant. To determine explicitly the entropic normalization additive constant  $b$ , we proceed as follows:

$$H_F(\theta) = - \int p_F(x; \theta) \log p_F(x; \theta) dx \quad (85)$$

$$= F(\theta) - \int p_F(x; \theta) \langle t(x), \theta \rangle dx - \int k(x) p_F(x; \theta) dx \quad (86)$$

$$= F(\theta) - \left\langle \int p_F(x; \theta) t(x) dx, \theta \right\rangle - \int k(x) p_F(x; \theta) dx \quad (87)$$

$$= H_F(\theta) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle + b \quad (88)$$

That is, the constant is given by  $b = - \int k(x) p_F(x; \theta) dx = -E_{\theta}[k(x)]$ . (It depends on the member  $\theta$  of the family for  $k(x) \neq 0$ .)  $\blacksquare$

**Example** Consider the Poisson distribution with probability mass function  $p(x; \lambda) = \frac{\lambda^x \exp(-\lambda)}{x!}$ . The canonical decomposition yields  $\theta = \log \lambda$ ,  $F(\theta) = \exp \theta = \lambda$  (derivative is  $F'(\theta) = \exp \theta = \lambda$ ),  $t(x) = x$  and  $k(x) = -\log x!$ . The Poisson entropy is therefore  $F(\theta) - \theta F'(\theta) + b = \lambda(1 - \log \lambda) - E[k(x)]$ . Since  $k(x) = -\log x!$ , we have  $b = -E[k(x)] = \sum_{k=0}^{\infty} p_F(x; \lambda) \log k! = e^{-\lambda} \sum \frac{\lambda^k \log k!}{k!}$ .  $\diamond$

#### V. SUMMARY, CONCLUSION AND DISCUSSION

In this paper, we have given closed-form expressions for the Rényi and Tsallis divergences of distributions  $p \sim E_F(\theta)$  and  $q \sim E_F(\theta')$  belonging to the same exponential family  $E_F$ :

$$D_{\alpha}^R(p : q) = \frac{J_{F,\alpha}(\theta : \theta')}{1-\alpha}, \quad (89)$$

$$D_{\alpha}^T(p : q) = \frac{e^{-J_{F,\alpha}(\theta : \theta')} - 1}{\alpha - 1}, \quad (90)$$

$$\text{KL}(p : q) = \lim_{\alpha \rightarrow 1} D_{\alpha}^R(p : q) \quad (91)$$

$$= \lim_{\alpha \rightarrow 1} D_{\alpha}^T(p : q) = B_F(\theta' : \theta), \quad (92)$$

where

$$\begin{aligned} J_{F,\alpha}(\theta : \theta') &= \alpha F(\theta) + (1-\alpha)F(\theta') - F(\alpha\theta + (1-\alpha)\theta') \\ &= J_{F,1-\alpha}(\theta' : \theta) \end{aligned} \quad (93)$$

is the skew Jensen divergence. Since the Rényi divergence for  $\alpha = \frac{1}{2}$  is related to the Bhattacharyya coefficient and Hellinger distance, this also yields closed-form expressions for members of the same exponential family:

$$B(p, q) = e^{-J_{F,\frac{1}{2}}(\theta, \theta')}, \quad (95)$$

$$H(p, q) = \sqrt{1 - e^{-J_{F,\frac{1}{2}}(\theta, \theta')}}. \quad (96)$$

Furthermore, we showed that the Rényi and Tsallis entropies, including Shannon entropy in the limit case, can be expressed respectively as

$$H_{\alpha}^R(p_F(x; \theta)) = \frac{1}{1-\alpha} \left( F(\alpha\theta) - \alpha F(\theta) + \log E_p[e^{(\alpha-1)k(x)}] \right) \quad (97)$$

$$H_{\alpha}^T(p_F(x; \theta)) = \frac{1}{1-\alpha} \left( (e^{F(\alpha\theta) - \alpha F(\theta)}) E_p[e^{(\alpha-1)k(x)}] - 1 \right) \quad (98)$$

$$H(p_F(x; \theta)) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle - E_p[k(x)] \quad (99)$$



The Shannon cross-entropy is given by

$$H^\times(p_F(x; \theta) : p_F(x; \theta')) = F(\theta') - \langle \theta', \nabla F(\theta) \rangle - E_p[k(x)] \quad (100)$$

Thus these entropies admit closed-form formula whenever the normalizing carrier measure is zero ( $k(x) = 0$ ):

$$H_\alpha^R(p_F(x; \theta)) = \frac{1}{1-\alpha} (F(\alpha\theta) - \alpha F(\theta)) \quad (101)$$

$$H_\alpha^T(p_F(x; \theta)) = \frac{1}{1-\alpha} (e^{F(\alpha\theta) - \alpha F(\theta)} - 1) \quad (102)$$

$$H(p_F(x; \theta)) = F(\theta) - \langle \theta, \nabla F(\theta) \rangle \quad (103)$$

This includes the case of Bernoulli, exponential, Gaussian and center Laplacian distributions, among others. (We report in A the Rényi entropy for multivariate Gaussian distributions using matrix calculus.)

Recently, Poczos and Schneider [20] have proposed a technique to estimate the  $\alpha$ -divergence based on the  $k$ -nearest neighbor graph. Although applicable to any kind of distributions, their method is computationally intensive and limited in practice to small dimensions. In contrast, we may estimate the Rényi entropy and divergence of distributions belonging to the same exponential family by applying the closed-form expressions on the estimates of parameter distributions. This is all the more efficient as the maximum likelihood estimator (MLE) of exponential families for independent and identically distributed (i.i.d.) observations  $x_1, \dots, x_n$  is also available in closed-form:

$$\hat{\theta} = (\nabla F)^{-1} \left( \frac{1}{n} \sum_{i=1}^n t(x_i) \right). \quad (104)$$

This estimate  $\hat{\theta}$  is termed the *observed point* in information geometry [12].

## REFERENCES

- [1] C. E. Shannon, "A mathematical theory of communication," *The Bell System Technical Journal*, vol. 27, pp. 379–423, 1948.
- [2] T. M. Cover and J. A. Thomas, *Elements of information theory*. New York, NY, USA: Wiley-Interscience, 1991.
- [3] A. Darbellay and I. Vajda, "Entropy Expressions for Multivariate Continuous Distributions," *IEEE Transactions on Information Theory*, vol. 46, no. 2, pp. 709–712, 2000.
- [4] A. Rényi, "On measures of entropy and information," in *Proceedings of the 4th Berkeley Symposium on Mathematical Statistics and Probabilities*, vol. 1, 1961, pp. 547–561.
- [5] K. Zografos and S. Nadarajah, "Expressions for Rényi and Shannon entropies for multivariate distributions," *Statistics & Probability Letters*, vol. 71, no. 1, pp. 71–84, 2005.
- [6] A. O. Hero, B. Ma, O. Michel, and J. D. Gorman, "Alpha-divergence for classification, indexing and retrieval," *Comm. and Sig. Proc. Lab. (CSPL)*, Dept. EECS, University of Michigan, Ann Arbor, Tech. Rep. 328, July, 2001.
- [7] J. Havrda and F. Chárvat, "Quantification method of classification processes," *Kybernetika Cista*, vol. 1, no. 3, pp. 30–34, 1967.
- [8] H. Suyari and M. Tsukada, "Tsallis differential entropy and divergences derived from the generalized Shannon-Khinchin axioms," in *Proceedings of the 2009 IEEE international conference on Symposium on Information Theory (ISIT)*, ser. ISIT'09, vol. 1. Piscataway, NJ, USA: IEEE Press, 2009, pp. 149–153.
- [9] O. Barndorff-Nielsen, *Information and exponential families in statistical theory*. Wiley, 1978.
- [10] F. Nielsen and V. Garcia, "Statistical exponential families: A digest with flash cards," 2009, arXiv.org:0911.4863.

- [11] H. Chernoff, "A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations," *Annals of Mathematical Statistics*, vol. 23, pp. 493–507, 1952.
- [12] S. Amari and H. Nagaoka, *Methods of Information Geometry*, A. M. Society, Ed. Oxford University Press, 2000.
- [13] I. Csizár, "Information-type measures of difference of probability distributions and indirect observation," *Studia Scientiarum Mathematicarum Hungarica*, vol. 2, pp. 229–318, 1967.
- [14] A. Bhattacharyya, "On a measure of divergence between two statistical populations defined by their probability distributions," *Bulletin of Calcutta Mathematical Society*, vol. 35, pp. 99–110, 1943.
- [15] E. D. Hellinger, "Die orthogonalinvarianten quadratischer formen von unendlich vielen variablen," 1907, thesis of the university of Göttingen.
- [16] T. van Erven and P. Harremoës, "Rényi divergence and its properties," *CoRR*, vol. abs/1001.4448, 2010.
- [17] C. R. Rao and T. K. Nayak, "Cross entropy, dissimilarity measures, and characterizations of quadratic entropy," *IEEE Transactions on Information Theory*, vol. 31, no. 5, pp. 589–593, sep 1985.
- [18] A. Banerjee, S. Merugu, I. S. Dhillon, and J. Ghosh, "Clustering with Bregman divergences," *Journal of Machine Learning Research*, vol. 6, pp. 1705–1749, 2005.
- [19] L. M. Bregman, "The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming," *USSR Computational Mathematics and Mathematical Physics*, vol. 7, pp. 200–217, 1967.
- [20] B. Poczos and J. Schneider, "On the estimation of  $\alpha$ -Divergences," in *Proceedings of the 14th International Conference on AI and Statistics (AISTATS)*, April 2011, volume 15 of *Journal of Machine Learning Research (JMLR)*.

## APPENDIX

Rényi and Tsallis entropies and divergences for multivariate Gaussians

The probability density of a multivariate Gaussian centered at  $\mu$  with covariance matrix  $\Sigma$  is given by

$$p(x; \mu, \Sigma) = \frac{1}{(2\pi)^{\frac{d}{2}} \sqrt{|\Sigma|}} \exp - \frac{(x - \mu)^T \Sigma^{-1} (x - \mu)}{2} \quad (105)$$

Let us rewrite this density to fit the canonical decomposition of exponential families:

$$\begin{aligned} p(x; \mu, \Sigma) &= \exp \left( -\frac{1}{2} x^T \Sigma^{-1} x + x \mu^T \Sigma^{-1} - \frac{1}{2} \mu^T \Sigma^{-1} \mu \right. \\ &\quad \left. - \frac{1}{2} \log(2\pi)^d |\Sigma| \right) \\ &= \exp \left( \left\langle (x, x^T x), \left( \Sigma^{-1} \mu, -\frac{1}{2} \Sigma^{-1} \right) \right\rangle - F(\theta) \right) \end{aligned} \quad (106)$$

with  $\theta = (\Sigma^{-1} \mu, -\frac{1}{2} \Sigma^{-1})$  and  $F(\theta) = \frac{1}{2} \log(2\pi)^d |\Sigma| + \frac{1}{2} \mu^T \Sigma^{-1} \mu$  (and  $k(x) = 0$ ). Natural parameter  $\theta = (\Sigma^{-1} \mu, -\frac{1}{2} \Sigma^{-1}) = (v, M)$  consists in two parts: a vectorial part  $v$ , and a symmetric negative definite matrix part  $M \preceq 0$ . The inner product of  $\theta = (v, M)$  and  $\theta' = (v', M')$  is defined as

$$\langle \theta, \theta' \rangle = v^T v' + \text{tr}(M^T M'), \quad (108)$$

where  $\text{tr}$  denote the matrix trace (i.e., sum of diagonal elements).

Since  $|\Sigma| = -\frac{1}{2|M|}$  ( $|M| = -\frac{1}{2}|\Sigma^{-1}| = -\frac{1}{2|\Sigma|}$ ) and  $\mu = \Sigma v = -\frac{1}{2} M^{-1} v$ , it follows that the log-normalizer expressed using the natural parameters is

$$F(\mu, \Sigma) = \frac{1}{2} \log(2\pi)^d |\Sigma| + \frac{1}{2} \mu^T \Sigma^{-1} \mu \quad (109)$$

$$F(v, M) = \frac{1}{2} \log(2\pi)^d + \frac{1}{2} \log \left( -\frac{1}{2|M|} \right) - \frac{1}{4} v^T M^{-1} v \quad (110)$$

Let us now write the term  $F(\alpha\theta)$ :

$$F(\alpha\theta) = F(\alpha v, \alpha M) \quad (111)$$

$$= \frac{1}{2} \log(2\pi)^d + \frac{1}{2} \log \left( -\frac{1}{2|\alpha M|} \right) - \frac{1}{4} \alpha v^T M^{-1} \alpha v \quad (112)$$

We shall use the fact that  $|\alpha M| = \alpha^d |M|$  for  $d$ -dimensional matrices. It follows<sup>4</sup> that

$$H_\alpha^R(\theta) = \frac{1}{1-\alpha} (F(\alpha\theta) - \alpha F(\theta)) \quad (113)$$

$$= \frac{1}{1-\alpha} \left( \frac{d}{2} (1-\alpha) \log 2\pi + \frac{1}{2} \log -\frac{1}{2|\alpha M|} \right. \quad (114)$$

$$\left. -\frac{\alpha}{2} \log -\frac{1}{2|M|} \right) \quad (115)$$

$$= \frac{d}{2} \log 2\pi + \frac{1}{1-\alpha} \left( \frac{1}{2} \log |\Sigma| - \frac{d}{2} \log \alpha - \frac{\alpha}{2} \log |\Sigma| \right) \quad (116)$$

$$= \frac{d}{2} \log 2\pi + \frac{1}{2} \log |\Sigma| - \frac{d \log \alpha}{2(1-\alpha)} \quad (117)$$

## APPENDIX

Kullback-Leibler divergence of exponential families as Bregman divergences

Let us prove that for two distributions  $p \sim E_F(\theta)$  and  $q \sim E_F(\theta')$  belonging to the same exponential family  $E_F$ , we have

$$\text{KL}(p : q) = B_F(\theta' : \theta) \quad (118)$$

*Proof:* We first show that  $\nabla F(\theta) = E[t(X)]$  with  $t(X)$  the sufficient statistics:

$$F(\theta) = \log \int_x \exp(\langle t(x), \theta \rangle + k(x)) dx \quad (119)$$

$$\nabla F(\theta) = \left[ \frac{\int_x t(x) \exp(\langle t(x), \theta \rangle + k(x)) dx}{\int_x \exp\{\langle t(x), \theta \rangle + k(x)\} dx} \right]_j \quad (120)$$

Since  $e^{F(\theta)} = \int_x \exp(\langle t(x), \theta \rangle + k(x)) dx$ , we replace the denominator to get

$$\nabla F(\theta) = \int_x t(x) \exp\{\langle t(x), \theta \rangle - F(\theta) + k(x)\} dx \quad (121)$$

$$= \int_x t(x) p_F(x; \theta) dx \quad (122)$$

$$= E_\theta[t(x)] \quad (123)$$

We are now ready to prove  $\text{KL}(p : q) = B_F(\theta' : \theta)$ :

<sup>4</sup>Note that the terms  $-\frac{1}{4}(\alpha v)^T (\alpha M)^{-1} (\alpha v) + \frac{1}{4} \alpha v^T M^{-1} v$  vanishes so that Rényi entropy does not depend on the mean parameter  $\mu$ .

$$\text{KL}(p : q) = \int_x p_F(x; \theta) \log \frac{p_F(x; \theta)}{p_F(x; \theta')} dx \quad (124)$$

$$= \int_x p_F(x; \theta) (F(\theta') - F(\theta) + \langle \theta - \theta', t(x) \rangle) dx \quad (125)$$

$$= \int_x p_F(x; \theta) (B_F(\theta' : \theta) + \langle \theta' - \theta, \nabla F(\theta) \rangle + \langle \theta - \theta', t(x) \rangle) dx \quad (126)$$

$$= B_F(\theta' : \theta) + \int_x p_F(x; \theta) \langle \theta' - \theta, \nabla F(\theta) - t(x) \rangle dx \quad (127)$$

$$= B_F(\theta' : \theta) - \int_x p_F(x; \theta) \langle \theta' - \theta, t(x) \rangle dx \quad (128)$$

$$+ \langle \theta' - \theta, \nabla F(\theta) \rangle \quad (129)$$

$$= B_F(\theta' : \theta) - \langle \theta' - \theta, \int_x p_F(x; \theta) t(x) dx \rangle \quad (130)$$

$$+ \langle \theta' - \theta, \nabla F(\theta) \rangle \quad (131)$$

$$= B_F(\theta' : \theta) \quad (132)$$

since  $\nabla F(\theta) = E[t(X)]$ . ■

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